

FORCE A SET MODEL OF Z_3 + HARRINGTON'S PRINCIPLE VIA SET FORCING

YONG CHENG

ABSTRACT. Let Z_3 denote 3^{rd} order arithmetic. Let Harrington's Principle, **HP**, denote the statement that there is a real x such that every x -admissible ordinal is a cardinal in L . In this paper, assuming there exists a remarkable cardinal with a weakly inaccessible cardinal above it, we force a set model of Z_3 + **HP** via set forcing without reshaping.

1. INTRODUCTION

Harrington proved in 1978 the following classical theorem which stimulates the research on the relationship between large cardinals and determinacy hypothesis since then.

Theorem 1.1. (*Harrington, [5]*) $(ZF) \quad Det(\Sigma_1^1) \text{ implies } 0^\sharp \text{ exists.}$

Definition 1.2. Let Harrington's Principle, **HP** for short, denote the following statement: $\exists x \in 2^\omega \forall \alpha (\alpha \text{ is } x\text{-admissible} \rightarrow \alpha \text{ is an } L\text{-cardinal})$.

Theorem 1.3. (*Silver, [5]*) $(ZF) \quad \text{HP implies } 0^\sharp \text{ exists.}$

Definition 1.4. (i) $Z_2 = ZFC^- + \text{Any set is Countable.}$ ¹

(ii) $Z_3 = ZFC^- + \mathcal{P}(\omega) \text{ exists} + \text{Any set is of cardinality } \leq \beth_1$.

(iii) $Z_4 = ZFC^- + \mathcal{P}(\mathcal{P}(\omega)) \text{ exists} + \text{Any set is of cardinality } \leq \beth_2$.

Z_2, Z_3 and Z_4 are the corresponding axiomatic systems for Second Order Arithmetic (SOA), Third Order Arithmetic and Fourth Order Arithmetic. Note that $Z_3 \vdash H_{\omega_1} \models Z_2, Z_4 \vdash H_{\beth_1^+} \models Z_3$ and " $\exists A \subseteq \omega_1 (V = L[A]) + Z_3$ " $\vdash \omega_1$ is the largest cardinal.

The known proofs of Theorem 1.1 are done in two steps: first show that $Det(\Sigma_1^1)$ implies **HP** and then show that **HP** implies 0^\sharp exists. We observe that the first step is provable in Z_2 . For the proof of " $Z_2 + Det(\Sigma_1^1)$ implies **HP**", see [3]. In this paper, we aim to prove the following main theorem.

The Main Theorem 1.5. Assuming there exists a remarkable cardinal with a weakly inaccessible cardinal above it, we can force a set model of Z_3 + **HP** via set forcing without reshaping.

2000 *Mathematics Subject Classification.* 03E35, 03E55, 03E30.

Key words and phrases. Harrington's Principle, 0^\sharp , almost disjoint forcing, Baumgartner's forcing, strong reflecting property, remarkable cardinal, Z_3 .

This paper is based on part of the author's Ph.D. thesis written in 2012 at the National University of Singapore under the supervision of Chong Chi Tat and W.Hugh Woodin. I would like to express my deep gratitude to W.Hugh Woodin for all his support and guidance on the thesis. I would like to thank members of my Ph.D. committee. I would like to thank Ralf Schindler for pointing out that Step One can be done by assuming weaker large cardinal: remarkable cardinal.

¹ ZFC^- denotes ZFC with the Power Set Axiom deleted.

2. DEFINITIONS AND PRELIMINARIES

Our definitions and notations are standard. We refer to standard textbooks as [8], [10] and [11] for the definitions and notations we use. For the definition of admissible set and admissible ordinal, see [1] and [4]. For notions of large cardinals, see [10]. Our notations about forcing are standard (see [7] and [8]). Almost disjoint forcing is standard (see [8] and [11]). We say that 0^\sharp exists if there exists an iterable premouse of the form (J_α, \in, U) where $U \neq \emptyset$. For the theory of 0^\sharp see e.g. [16]. We can define 0^\sharp in Z_2 . In Z_2 , 0^\sharp exists if and only if $\exists x \in \omega^\omega$ (x codes a countable iterable premouse) which is a Σ_3^1 statement.

Note that under $V = L$, $H_\eta = L_\eta$ for any L -cardinal η . In this paper, we often use H_η and L_η interchangeably. Throughout this paper whenever we write $X \prec H_\kappa$ and $\gamma \in X$, $\bar{\gamma}$ always denotes the image of γ under the transitive collapse of X .

Definition 2.1. (Ralf Schindler, [15])

- (1) A cardinal κ is remarkable if and only if for all regular cardinal $\theta > \kappa$ there are $\pi, M, \bar{\kappa}, \sigma, N$ and $\bar{\theta}$ such that the following hold: $\pi : M \rightarrow H_\theta$ is an elementary embedding, M is countable and transitive, $\pi(\bar{\kappa}) = \kappa$, $\sigma : M \rightarrow N$ is an elementary embedding with critical point $\bar{\kappa}$, N is countable and transitive, $\bar{\theta} = M \cap \text{Ord}$ is a regular cardinal in N , $\sigma(\bar{\kappa}) > \bar{\theta}$ and $M = H_{\bar{\theta}}^N$, i.e. $M \in N$ and $N \models M$ is the set of all sets which are hereditarily smaller than $\bar{\theta}$.
- (2) Let κ be a cardinal, G be $\text{Col}(\omega, < \kappa)$ -generic over V , $\theta > \kappa$ be a regular cardinal and $X \in [H_\theta^{V[G]}]^\omega$. We say that X condense remarkably if $X = \text{ran}(\pi)$ for some elementary $\pi : (H_\beta^{V[G \cap H_\alpha^V]}, \in, H_\beta^V, G \cap H_\alpha^V) \rightarrow (H_\theta^{V[G]}, \in, H_\theta^V, G)$ where $\alpha = \text{crit}(\pi) < \beta < \kappa$ and β is a regular cardinal in V .

Lemma 2.2. (Ralf Schindler, [15]) *A cardinal κ is remarkable if and only if for all regular cardinal $\theta > \kappa$ we have $\Vdash_{\text{Col}(\omega, < \kappa)}^V \{X \in [H_\theta^{V[G]}]^\omega : X \text{ condense remarkably}\}$ is stationary.*

Lemma 2.3. *Suppose κ is an L -cardinal. The followings are equivalent:*

- (1) κ is remarkable in L ;
- (2) If $\gamma \geq \kappa$ is an L -cardinal, $\theta > \gamma$ is a regular cardinal in L , then $\Vdash_{\text{Col}(\omega, < \kappa)}^L \{X | X \prec L_\theta[\dot{G}], |X| = \omega \wedge \bar{\gamma} \in X \wedge \bar{\gamma} \text{ is an } L\text{-cardinal}\}$ is stationary.

Proof. By Lemma 2.2, κ is remarkable in L iff if $\theta > \kappa$ is a regular cardinal in L and G is $\text{Col}(\omega, < \kappa)$ -generic over L , then $L[G] \models \{X \in [L_\theta[G]]^\omega | X = \text{ran}(\pi) \text{ for some elementary } \pi : (L_\beta[G \restriction \alpha], \in, L_\beta, G \restriction \alpha) \rightarrow (L_\theta[G], \in, L_\theta, G) \text{ where } \alpha = \text{crit}(\pi) < \beta < \kappa \text{ and } \beta \text{ is a regular cardinal in } L\}$ is stationary" iff if $\gamma \geq \kappa$ is an L -cardinal, $\theta > \gamma$ is a regular cardinal in L and G is $\text{Col}(\omega, < \kappa)$ -generic over L , then $L[G] \models \{X | X \prec L_\theta \wedge |X| = \omega \wedge \gamma \in X \wedge \bar{\gamma} \text{ is an } L\text{-cardinal}\}$ is stationary". \square

In the rest of this section, we assume that S is a stationary subset of ω_1 .

Definition 2.4. (Harrington's forcing, [6]) $P_S = \{p : p \text{ is a closed bounded subset of } \omega_1 \text{ and } p \subseteq S\}$. For $p, q \in P_S$, $p \leq q$ if and only if $p \supseteq q$ and for any $\alpha \in p \setminus q$, $\alpha > \sup(q)$.²

² $|P_S| = 2^\omega$, P_S is ω -distributive and hence assuming CH , P_S preserves all cardinals.

Definition 2.5. (Baumgartner's forcing, [2]) Define $P_S^B = \{f : \text{dom}(f) \rightarrow S \mid \text{dom}(f) \subseteq \omega_1 \text{ is finite and } \exists \alpha > \max(\text{dom}(f)) \exists g : \alpha \rightarrow S (g \text{ is continuous, increasing and } g \upharpoonright \text{dom}(f) = f)\}$. For $f, g \in P_S^B$, $g \leq f$ if and only if $f \subseteq g$.

Note that the following are equivalent: (1) $f \in P_S^B$; (2) $\text{dom}(f) \subseteq \omega_1$ is finite and there exists $g : \max(\text{dom}(f)) + 1 \rightarrow S$ such that g is continuous, increasing and $g \upharpoonright \text{dom}(f) = f$; (3) $\text{dom}(f) \subseteq \omega_1$ is finite and there exists $C \subseteq S$ such that C is closed, $\text{o.t.}(C) = \max(\text{dom}(f)) + 1$ and for any $\beta \in \text{dom}(f)$, $f(\beta)$ is the β -th element of C .

Let G be P_S^B -generic over V . Define $F_G = \bigcup \{f \mid f \in G\}$. Then $F_G : \omega_1 \rightarrow S$ is increasing, continuous and $\text{ran}(F_G)$ is a club in ω_1 .

Fact 2.6. (Baumgartner, [2]) $(Z_3) \quad |P_S^B| = \omega_1$ even not assuming CH and P_S^B preserves ω_1 .

Since P_S^B is ω_2 -c.c and preserves ω_1 , P_S^B preserves all cardinals.

Lemma 2.7. Suppose $f \in P_S^B$. Then $f \Vdash_{P_S^B} \dot{G} \cap (P_S^B)_f$ is $(P_S^B)_f$ -generic over V .

Proof. Suppose $h \in P_S^B$, $h \leq f$ and D is a dense subset of $(P_S^B)_f$. It suffices to show that there is $p \in D$ such that $h \cup p \in P_S^B$. Let $\max(\text{dom}(f)) = \beta$. Then $h \upharpoonright (\beta + 1) \in (P_S^B)_f$. Take $p \in D$ such that $p \leq h \upharpoonright (\beta + 1)$. We show that $h \cup p \in P_S^B$.

Let $\alpha = \max(\text{dom}(h))$. Since $h \in P_S^B$, there exists $E \subseteq S$ such that E is closed, $\text{o.t.}(E) = \alpha + 1$ and for any $\gamma \in \text{dom}(h)$, $h(\gamma)$ is the γ -th element of E . Since $p \in (P_S^B)_f$, $\max(\text{dom}(p)) = \beta$. Let $F \subseteq S$ be closed such that $\text{o.t.}(F) = \beta + 1$ and for any $\gamma \in \text{dom}(p)$, $p(\gamma)$ is the γ -th element of F . Note that $h(\beta) = f(\beta) = p(\beta)$. Let $C = \{\gamma \in E \mid \gamma \geq h(\beta) = p(\beta)\} \cup F$. $C \subseteq S$ is closed. Since $p \leq h \upharpoonright (\beta + 1)$, $\text{o.t.}(C) = \alpha + 1$. For any $\gamma \in \text{dom}(h \cup p)$, $(h \cup p)(\gamma)$ is the γ -th element of C . So $h \cup p \in P_S^B$. \square

Proposition 2.8. Suppose $\gamma \geq \omega_1$ is an L -cardinal. Then the following are equivalent:

- (1) For some regular cardinal $\kappa > \gamma$, $\forall X((X \prec H_\kappa, |X| = \omega \text{ and } \gamma \in X) \rightarrow \bar{\gamma} \text{ is an } L\text{-cardinal})$.
- (2) There exists $F : \gamma^{<\omega} \rightarrow \gamma$ such that if $X \subseteq \gamma$ is countable and closed under F , then $\text{o.t.}(X)$ is an L -cardinal.³
- (3) For any regular cardinal $\kappa > \gamma$, $\forall X((X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X) \rightarrow \bar{\gamma} \text{ is an } L\text{-cardinal})$.

Proof. (1) \Rightarrow (2) Let $\kappa > \gamma$ be the witness regular cardinal for (1). Let $Z = \{X \mid X \prec H_\kappa, |X| = \omega, \gamma \in X \text{ and } \bar{\gamma} \text{ is an } L\text{-cardinal}\}$. Then $Z \upharpoonright \gamma = \{X \cap \gamma \mid X \in Z\}$ contains a club E in $[\gamma]^\omega$. So there exists $F : \gamma^{<\omega} \rightarrow \gamma$ such that if $X \subseteq \gamma$ is countable and closed under F , then $X \in E$. Suppose $X \subseteq \gamma$ is countable and closed under F . We show that $\text{o.t.}(X)$ is an L -cardinal. Since $X \in E$, $X = Y \cap \gamma$ for some $Y \in Z$ and hence $\bar{\gamma}$ is an L -cardinal. So $\text{o.t.}(X) = \text{o.t.}(Y \cap \gamma) = \bar{\gamma}$ is an L -cardinal.

(2) \Rightarrow (3) Suppose $F : \gamma^{<\omega} \rightarrow \gamma$ has the property that

(2.1) if $X \subseteq \gamma$ is countable and closed under F , then $\text{o.t.}(X)$ is an L -cardinal.

³In this paper we say X is closed under F if $F''X^{<\omega} \subseteq X$.

Suppose $\kappa > \gamma$ is regular, $X \prec H_\kappa$, $|X| = \omega$ and $\gamma \in X$. We show that $\bar{\gamma}$ is an L -cardinal. Note that $F \in H_\kappa$ and the property of F is definable in H_κ . Since $\gamma \in X$, $F \in X$ and the property of F is definable in X . So $X \cap \gamma$ is closed under F . By (2.1), $\text{o.t.}(X \cap \gamma)$ is an L -cardinal. But $\bar{\gamma} = \text{o.t.}(X \cap \gamma)$. \square

Definition 2.9. Let $\gamma \geq \omega_1$ be an L -cardinal. We say γ has strong reflecting property if Proposition 2.8(1) holds.

Proposition 2.10. Suppose $\gamma \geq \omega_1$ is an L -cardinal and $|\gamma| = \omega_1$. Then the following are equivalent:

- (a) γ has strong reflecting property.
- (b) For any bijection $\pi : \omega_1 \rightarrow \gamma$, there exists a club $D \subseteq \omega_1$ such that for any $\theta \in D$, $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L -cardinal.
- (c) For some bijection $\pi : \omega_1 \rightarrow \gamma$, there exists a club $D \subseteq \omega_1$ such that for any $\theta \in D$, $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L -cardinal.

Proof. (a) \Rightarrow (b) Let $\kappa > \gamma$ be the regular cardinal that witnesses the strong reflecting property of γ . Let $E = \{X \cap \omega_1 \mid X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X\}$. Then E contains a club D in ω_1 . Let $\pi : \omega_1 \rightarrow \gamma$ be a bijection and $\beta \in D$. So $\beta = X \cap \omega_1 \in E$ for some X . Note that $\bar{\gamma} = \text{o.t.}(\{\pi(\alpha) \mid \alpha < X \cap \omega_1\})$. So $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \beta\}) = \bar{\gamma}$ is an L -cardinal.

(b) \Rightarrow (a) Let $\kappa > \gamma$ be a regular cardinal with $\kappa \geq (2^{\omega_1})^+$. Suppose $X \prec H_\kappa$, $|X| = \omega$ and $\gamma \in X$. We show that $\bar{\gamma}$ is an L -cardinal. Let $\pi : \omega_1 \rightarrow \gamma$ be the witness bijection for $|\gamma| = \omega_1$ and $D \subseteq \omega_1$ be the witness club for π by (b). Note that π and D are first order definable in H_κ . So $\pi, D \in X$. Since D is unbounded in $X \cap \omega_1$, $X \cap \omega_1 \in D$. Note that $\bar{\gamma} = \text{o.t.}(\{\pi(\alpha) \mid \alpha \in X \cap \omega_1\})$. So $\bar{\gamma}$ is an L -cardinal.

(c) \Rightarrow (a) follows by the similar argument as (b) \Rightarrow (a). \square

Let (1)*, (2)* and (3)* respectively denote the statements which replace “is an L -cardinal” with “is not an L -cardinal” in Proposition 2.8(1), Proposition 2.8(3) and Proposition 2.10(b). The following corollary is an observation from proofs of Proposition 2.8 and Proposition 2.10.

Corollary 2.11. Suppose $\gamma \geq \omega_1$ is an L -cardinal and $|\gamma| = \omega_1$. Then (2)* \Leftrightarrow (1)* \Leftrightarrow (3)*.

Proposition 2.12. Suppose $\gamma \geq \omega_1$ is an L -cardinal. The statement “ γ has strong reflecting property” is upward absolute.

Proof. Suppose $M \subseteq N$ and $M \models \gamma$ has strong reflecting property. We show that $N \models \gamma$ has strong reflecting property.

By Proposition 2.8, in M , there exists $F : \gamma^{<\omega} \rightarrow \gamma$ such that (2.1) holds. By Proposition 2.8, it suffices to show that in N , (2.1) holds.

Suppose not. Then in N , there exists $\bar{\gamma} < \omega_1$ such that $\bar{\gamma}$ is not an L -cardinal and there exists an order preserving $j : \bar{\gamma} \rightarrow \gamma$ such that $\text{ran}(j)$ is closed under F . So in N , there exists $e : \omega \rightarrow L_{\omega_1^N}$ and $\gamma' \in e''\omega$ such that $e''\omega \prec L_{\omega_1^N}, L_{\omega_1^N} \models \text{“}\gamma' \text{ is not an } L\text{-cardinal”}$ and there exists an order preserving $j' : \text{o.t.}(e''\omega \cap \gamma') \rightarrow \gamma$ such that $\text{ran}(j')$ is closed under F .

Let $\langle \varphi_i \mid i \in \omega \rangle$ be a recursive enumeration of formulas with infinite repetitions. We assume that for $i \in \omega$, φ_i has free variables among x_0, \dots, x_{i+1} . So in N , there exist $e : \omega \rightarrow L_{\omega_1^N}$, $\pi : \omega \rightarrow \gamma$ and $\gamma^* \in e''\omega$ such that (1) for any $i \in \omega$, if

there exists $a \in L_{\omega_1^N}$ such that $L_{\omega_1^N} \models \varphi_i[a, e(0), \dots, e(i)]$, then $L_{\omega_1^N} \models \varphi_i[e(2i+1), e(0), \dots, e(i)]$; (2) $\text{ran}(\pi)$ is closed under F ; (3) $L_{\omega_1^N} \models \gamma^*$ is not an L -cardinal; and (4) for $i \in \omega$, if $e(i) \notin \gamma^*$, then $\pi(i) = 0$; for $i < j \in \omega$, if $e(i), e(j) \in \gamma^*$, then $\pi(i) < \pi(j) \Leftrightarrow e(i) < e(j)$ and $\pi(i) = \pi(j) \Leftrightarrow e(i) = e(j)$. In N , let $T = \{(e \upharpoonright n, \pi \upharpoonright n) : e \text{ and } \pi \text{ have properties (1) – (4)}\}$. T is a tree and from (1) – (4), by absoluteness, $T \in M$. Since in N , there exists (e, π) satisfying (1) – (4), T has an infinite branch in N . By absoluteness, T has an infinite branch in M and such a branch corresponds to the existence of (e, π) with properties (1) – (4) in M . So in M , there exists $X \subseteq \gamma$ such that X is countable, closed under F and $\text{o.t.}(X)$ is not an L -cardinal which contradicts (2.1). \square

3. PROOF OF THE MAIN THEOREM

In this section we assume that there exists a remarkable cardinal with a weakly inaccessible cardinal above it. We want to force a set model of $Z_3 + \text{HP}$ via set forcing without reshaping.

We give an outline of the proof of our main Theorem 1.5. In Step One, we force over L to get a club in ω_2 of L -cardinals with strong reflecting property. This is necessary to show in Step Two that (3.3) holds. In Step Two, we force to get $A \subseteq \omega_1$ such that (3.3) holds in the generic extension. (3.3) motivates the definition of S and is necessary to show that S as defined in (3.14) contains a club in ω_1 and hence is stationary. In Step Three, we shoot a club C through S via Baumgartner's forcing such that (3.30) holds which will be used to define the almost disjoint system and show that the generic real via almost disjoint forcing satisfies **HP**. In Step Four, we use properties of $\text{Lim}(C)$ (Lemma 3.10 and Lemma 3.18) to define the almost disjoint system on ω and some $B^* \subseteq \omega_1$. Then we do almost disjoint forcing to code B^* by a real x . Finally, we use properties of $\text{Lim}(C)$ ((3.28), (3.29) and (3.30)) to show that x is the witness real for **HP**.

3.1. Step One. In this step we force over L to get a club in ω_2 of L -cardinals with strong reflecting property.

We work in L . Let κ be a remarkable cardinal and $\lambda > \kappa$ be an inaccessible cardinal. Suppose \bar{G} is $\text{Col}(\omega, < \kappa)$ -generic over L and G is $\text{Col}(\omega, < \kappa) * \text{Col}(\kappa, < \lambda)$ -generic over L . Now we work in $L[G]$.

Define $K = \{\gamma \mid \omega_1 \leq \gamma < \omega_2 \wedge \gamma \text{ is an } L\text{-cardinal}\}$.

Definition 3.1. For $\gamma \in K$, we say γ has weakly reflecting property if for some bijection $\pi : \omega_1 \rightarrow \gamma$, there exists stationary $D \subseteq \omega_1$ such that for any $\theta \in D$, $\text{o.t.}(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L -cardinal.

Proposition 3.2. $L[G] \models \text{for any } \gamma \in K, \gamma \text{ has weakly reflecting property.}$

Proof. We work in $L[G]$. Suppose $\gamma \in K$ is a counterexample and $\theta > \gamma$ is a regular cardinal. Since κ is remarkable in L , by Lemma 2.3, $L[\bar{G}] \models \{X \mid X \prec H_\theta \wedge |X| = \omega \wedge \gamma \in X \wedge \bar{\gamma} \text{ is an } L\text{-cardinal}\}$ is stationary. Note that the property “ $X \prec H_\theta \wedge |X| = \omega \wedge \gamma \in X \wedge \bar{\gamma} \text{ is an } L\text{-cardinal}$ ” is absolute between $L[\bar{G}]$ and $L[G]$. So by absoluteness, in $L[G]$,

$$(3.1) \quad \exists X (X \prec H_\theta \wedge |X| = \omega \wedge \gamma \in X \wedge \bar{\gamma} \text{ is an } L\text{-cardinal}).$$

Since γ does not have weakly reflecting property, (3)* in Corollary 2.11 holds and hence, by Corollary 2.11, (2)* holds which contradicts (3.1). \square

So K is a club in ω_2 of L -cardinals with weakly reflecting property. For $\gamma \in K$, by Proposition 3.2, there exist a bijection $\pi : \omega_1 \leftrightarrow \gamma$ and a stationary set $S \subseteq \omega_1$ such that for any $\theta \in S$, o.t. $(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L -cardinal (let π_γ and S_γ be such π and S). Then S_γ is stationary for $\gamma \in K$.

Definition 3.3. Suppose κ is a regular cardinal and $\{P_i : i \in I\}$ is a collection of partial orders. The κ -product of $\{P_i : i \in I\}$ is defined as $P = \{p : \text{dom}(p) = I \wedge \forall i \in I (p(i) \in P_i) \wedge |\text{suppt}(p)| < \kappa\}$ where $\text{suppt}(p) = \{i \in I : p(i) \neq 1_{P_i}\}$.

Let P be the ω_1 -product of $\{P_{S_\gamma} : \gamma \in K\}$ where P_{S_γ} is Harrington forcing. Since CH holds in $L[G]$, $|P_{S_\gamma}| = \omega_1$ for $\gamma \in K$.

Fact 3.4. ([8]) Assume $\kappa^{<\kappa} = \kappa$. If for every $i \in I$, $|P_i| \leq \kappa$, then the κ -product of P_i satisfies κ^+ -c.c.

In $L[G]$, $\omega_1^{<\omega_1} = \omega_1$. By Fact 3.4, P is ω_2 -c.c. For $\gamma \in K$, P_{S_γ} is ω -distributive and hence preserves ω_1 .

Lemma 3.5. P is ω -distributive.

Proof. It suffices to show that if $p \Vdash \dot{f} : \omega \rightarrow \text{Ord}$, then $\exists q \leq p \exists g(q \Vdash \dot{f} = \check{g})$. Suppose $p \Vdash \dot{f} : \omega \rightarrow \text{Ord}$. By induction on α we construct a chain $\{A_\alpha : \alpha < \omega_1\}$ of countable subsets of P .

Let $A_0 = \{p\}$. If α is limit, let $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$. For $\gamma \in K$ and $\alpha < \omega_1$, define $\theta_\alpha^\gamma = \sup(\{\sup(q(\gamma)) : q \in A_\alpha\})$. Given A_α we define $A_{\alpha+1}$ as follows. Note that for any $p \in A_\alpha$ and $n \in \omega$, there is $q = q(p, n) \in P$ such that $q \leq p$, q decides $\dot{f}(n)$ and $\forall \gamma \in \text{suppt}(q) (\sup(q(\gamma)) > \theta_\alpha^\gamma)$. Let $A_{\alpha+1} = A_\alpha \cup \{q(p, n) \mid p \in A_\alpha, n \in \omega\}$.

From our definitions, for $\gamma \in K$, $\langle \theta_\alpha^\gamma : \alpha < \omega_1 \rangle$ is increasing and continuous. For $\gamma \in K$, define $C_\gamma = \{\eta < \omega_1 : \alpha < \eta \rightarrow \theta_\alpha^\gamma < \eta\}$. Then C_γ is a club. Since S_γ is stationary, there is $\eta_\gamma \in S_\gamma \cap C_\gamma$ such that η_γ is a limit point of C_γ . For $\gamma \in K$, pick a sequence $\langle \alpha_n^\gamma : n \in \omega \rangle$ from C_γ such that $\lim_{n \in \omega} \alpha_n^\gamma = \eta_\gamma$. Then $\lim_{n \in \omega} \theta_{\alpha_n^\gamma}^\gamma = \eta_\gamma$.

Now we construct a sequence $\langle p_n : n \in \omega \rangle$ by induction. Let $p_0 = p$, $\beta_0 = 0$ and $s_0 = \text{suppt}(p_0)$. Suppose we have defined p_n, β_n and s_n . Let $\beta_{n+1} = \min\{\alpha_{n+1}^\gamma : \gamma \in s_n\}$. Take $p_{n+1} \in A_{\beta_{n+1}}$ such that $p_{n+1} \leq p_n$ and p_{n+1} decides $\dot{f}(n)$. WLOG, we can assume that $\langle \beta_n : n \in \omega \rangle$ is increasing since we only need that there are enough $\gamma \in s_n$ such that $p_{n+1} \in A_{\alpha_{n+1}^\gamma}$. Let $s_{n+1} = \text{suppt}(p_{n+1})$. Let $s = \bigcup_n s_n$. Then s is at most countable. Note that for any $\gamma \in s$ there is $N \in \omega$ such that for all $n \geq N$, $p_{n+1} \in A_{\alpha_{n+1}^\gamma}$. So $\forall \gamma \in s \exists N \forall n \geq N (\theta_{\alpha_n^\gamma}^\gamma < \sup(p_{n+1}(\gamma)) \leq \theta_{\alpha_{n+1}^\gamma}^\gamma)$. Hence for any $\gamma \in s$, $\lim_{n \in \omega} \sup(p_n(\gamma)) = \eta_\gamma$ since $\lim_{n \in \omega} \theta_{\alpha_n^\gamma}^\gamma = \eta_\gamma$.

Now we define the q we want as: if $\gamma \in s$, then $q(\gamma) = \bigcup_n p_n(\gamma) \cup \{\eta_\gamma\}$; otherwise, let $q(\gamma) = 1_{P_{S_\gamma}}$. For $\gamma \in s$, since $\eta_\gamma \in S_\gamma$, $q(\gamma)$ is a closed bounded subset of S_γ and hence $q(\gamma) \in P_{S_\gamma}$. So $q \in P$. Since $\forall n \in \omega (q \leq p_n \wedge p_{n+1}$ decides $\dot{f}(n))$, q decides $\dot{f}(n)$ for any $n \in \omega$. Define g as: $g(n) = \dot{f}(n)$. Then $q \leq p$ and $q \Vdash \dot{f} = \check{g}$. \square

So P preserves ω_1 and hence P preserves all cardinals. Let H be P -generic over $L[G]$. Now we work in $L[G, H]$. By (a) \Leftrightarrow (c) in Proposition 2.10,

$$(3.2) \quad L[G, H] \models \text{Any } \alpha \in K \text{ has strong reflecting property.}$$

So K is a club in ω_2 of L -cardinals with strong reflecting property.

3.2. Step Two. In this step, we first force over $L[G, H]$ to get $A \subseteq \omega_1$ such that $L[G, H, A] \models (3.3)$. Then we define S and show that S contains a club.

We still work in $L[G, H]$. Note that GCH holds. Take $B \subseteq \omega_2$ such that (1) $\omega^\omega \subseteq L[B]$; (2) $K \in L[B]$; (3) $\omega_2^{L[B]} = \omega_2$; and (4) $(L_{\omega_2}[B], K) \prec (H_{\omega_2}, K)$.

Now we work in $L[B]$. There exists a canonical sequence $\langle \delta_\alpha^* \mid \alpha < \omega_2 \rangle$ of pairwise almost disjoint subset of ω_1 such that δ_α^* is the $<_{L[B]}$ -least subset of ω_1 which is almost disjoint from any member of $\{\delta_\beta^* \mid \beta < \alpha\}$. By almost disjoint forcing, we get $A_0 \subseteq \omega_1$ such that $\alpha \in B \Leftrightarrow |A_0 \cap \delta_\alpha^*| < \omega_1$. The forcing preserves all cardinals.

Now we work in $L[A_0]$. Let $E = K \cap \{\eta \mid L_\eta[A_0] \prec L_{\omega_2}[A_0]\}$. Define $F : \mathcal{P}(\omega_1) \rightarrow \mathcal{P}(\omega_1)$ as follows: If $y \subseteq \omega_1$ codes γ , then $F(y) \subseteq \omega_1$ codes β where β is the least element of E such that $\beta > \gamma$ and $(L_\beta[A_0], E \cap \beta) \prec (L_{\omega_2}[A_0], E)$ (since E is a club in ω_2 , such β exists); If y does not code an ordinal, let $F(y) = \emptyset$.

There exists a canonical sequence $\langle \delta_\alpha \mid \alpha < \omega_2 \rangle$ of pairwise almost disjoint subset of ω_1 such that δ_α is the $<_{L[A_0, E]}$ -least subset of ω_1 which is almost disjoint from any member of $\{\delta_\beta \mid \beta < \alpha\}$. Let $\langle x_\alpha \mid \alpha < \omega_2 \rangle$ be the enumeration of $\mathcal{P}(\omega_1)$ in $L[A_0, E]$ in the order of construction. Define

$$Z_F = \{\alpha \cdot \omega_1 + \beta \mid \alpha < \omega_2 \wedge \beta \in F(x_\alpha)\}.$$

By almost disjoint forcing, we get $A_1 \subseteq \omega_1$ such that $\beta \in Z_F \Leftrightarrow |A_1 \cap \delta_\beta| < \omega_1$. Let $A = (A_0, A_1)$. The forcing preserves all cardinals.

Now we work in $L[G, H, A]$. Let α_A be the least α such that $L_\alpha[A] \models Z_3$. Note that $\omega_1 < \alpha_A < \omega_2$. We show that in $L[G, H, A]$,

(3.3)

if $\omega_1 \leq \alpha < \alpha_A$ is A -admissible, then α is an L -cardinal with strong reflecting property.

By (3.2) and Proposition 2.12, $L[G, H, A] \models \omega_1$ has strong reflecting property. Suppose $\omega_1 < \alpha < \alpha_A$ is A -admissible. Define

$$(3.4) \quad \gamma_0 = \sup(\{\gamma < \alpha \mid (L_\gamma[A_0], E \cap \gamma) \prec (L_{\omega_2}[A_0], E)\}).$$

If there is no $\gamma < \alpha$ such that $(L_\gamma[A_0], E \cap \gamma) \prec (L_{\omega_2}[A_0], E)$, let $\gamma_0 = 0$. Note that if $\gamma < \omega_2$ and $(L_\gamma[A_0], E \cap \gamma) \prec (L_{\omega_2}[A_0], E)$, then $\gamma \in E$. So if $\gamma_0 > 0$, then $\gamma_0 \in E$ and

$$(3.5) \quad (L_{\gamma_0}[A_0], E \cap \gamma_0) \prec (L_{\omega_2}[A_0], E).$$

Since E is a definable subset of $L_{\omega_2}[A_0]$, by (3.5), $E \cap \gamma_0$ is a definable subset of $L_{\gamma_0}[A_0]$ and hence

$$(3.6) \quad L_{\gamma_0}[A_0] = L_{\gamma_0}[A_0, E].$$

We assume that $\gamma_0 < \alpha$ and try to get a contradiction. Let α_0 be the least A_0 -admissible ordinal such that $\alpha_0 > \gamma_0$ and $\alpha_0 > \omega_1$. Since α is A_0 -admissible, $\alpha_0 \leq \alpha$.

Claim 3.6. $E \cap \alpha_0 = E \cap (\gamma_0 + 1)$.

Proof. We show that $E \cap \alpha_0 \subseteq E \cap (\gamma_0 + 1)$. Suppose $\gamma \in E \cap \alpha_0$ and $\gamma > \gamma_0$. Since $\gamma \in E$, $L_\gamma[A_0] \prec L_{\omega_2}[A_0]$. Since α_0 is definable from γ_0 and A_0 , α_0 is definable in $L_\gamma[A_0]$. So $\alpha_0 \leq \gamma$. Contradiction. \square

By Claim 3.6, $L_{\alpha_0}[A_0, E] = L_{\alpha_0}[A_0, E \cap \gamma_0]$.

Claim 3.7. $L_{\alpha_0}[A_0, E \cap \gamma_0] \models \gamma_0 < \omega_2$.

Proof. Suppose not. Then we have

$$(3.7) \quad \gamma_0 = \omega_2^{L_{\alpha_0}[A_0, E \cap \gamma_0]}.$$

Let P be the partial order which codes Z_F via $\langle \delta_\beta \mid \beta < \omega_2 \rangle$.⁴ From our definitions of E, F and $\langle x_\alpha \mid \alpha < \omega_2 \rangle$, P is a definable subset of $L_{\omega_2}[A_0]$. Standard argument gives that $(L_{\omega_2}[A_0], P) \models P$ is ω_2 -c.c. Let $P^* = P \cap L_{\gamma_0}[A_0]$. By (3.5), P^* is a definable subset of $L_{\gamma_0}[A_0]$ and

$$(3.8) \quad (L_{\gamma_0}[A_0], P^*) \models P^* \text{ is } \omega_2\text{-c.c.}$$

By (3.7), we have

$$(3.9) \quad L_{\alpha_0}[A_0, E \cap \gamma_0] \cap 2^{\omega_1} = L_{\gamma_0}[A_0, E \cap \gamma_0] \cap 2^{\omega_1}.$$

By (3.8), (3.6) and (3.9),

$$(3.10) \quad P^* \text{ is } \omega_2\text{-c.c. in } L_{\alpha_0}[A_0, E \cap \gamma_0].$$

We show that A_1 is generic over $L_{\alpha_0}[A_0, E \cap \gamma_0]$ for P^* . Let $Y \subseteq P^*$ be a maximal antichain with $Y \in L_{\alpha_0}[A_0, E \cap \gamma_0]$. By (3.7), (3.10) and (3.6), $Y \in L_{\gamma_0}[A_0, E \cap \gamma_0] = L_{\gamma_0}[A_0]$. By (3.5), Y is a maximal antichain in P . So the filter given by A_1 meets Y .

Since $\gamma_0 = \omega_2^{L_{\alpha_0}[A_0, E \cap \gamma_0]} = \omega_2^{L_{\alpha_0}[A_0, E \cap \gamma_0][A_1]}$ and by (3.6) $L_{\gamma_0}[A_0, E \cap \gamma_0][A_1] = L_{\gamma_0}[A]$, we have $L_{\gamma_0}[A] \models Z_3$ which contradicts the minimality of α_A . \square

From our definitions, we have

$$(3.11) \quad \text{for } \eta < \alpha_0, \langle \delta_\beta : \beta < \eta \rangle \in L_{\alpha_0}[A_0, E] = L_{\alpha_0}[A_0, E \cap \gamma_0] \text{ and}$$

$$(3.12) \quad \langle x_\beta \mid \beta < \alpha_0 \rangle \text{ enumerates } \mathcal{P}(\omega_1) \cap L_{\alpha_0}[A_0, E] = \mathcal{P}(\omega_1) \cap L_{\alpha_0}[A_0, E \cap \gamma_0].$$

Claim 3.8. If $y \subseteq \omega_1$ and $y \in L_{\alpha_0}[A_0, E \cap \gamma_0]$, then $F(y) \in L_{\alpha_0}[A]$.

Proof. Suppose $y \in \mathcal{P}(\omega_1) \cap L_{\alpha_0}[A_0, E \cap \gamma_0]$. By (3.12), $y = x_\xi$ for some $\xi < \alpha_0$. Note that $\xi \cdot \omega_1 + \alpha < \alpha_0$ for $\alpha < \omega_1$. Then $\alpha \in F(y)$ iff $\xi \cdot \omega_1 + \alpha \in Z_F$ iff $|A_1 \cap \delta_{\xi \cdot \omega_1 + \alpha}| < \omega_1$. By (3.11), $F(y) \in L_{\alpha_0}[A_0, E \cap \gamma_0][A_1]$. Since $E \cap \gamma_0 \in L_{\gamma_0+1}[A_0]$, $L_{\alpha_0}[A_0, E \cap \gamma_0][A_1] = L_{\alpha_0}[A]$. Hence $F(y) \in L_{\alpha_0}[A]$. \square

By Claim 3.7, there exists $y \in L_{\alpha_0}[A_0, E \cap \gamma_0] \cap \mathcal{P}(\omega_1)$ such that y codes γ_0 . By the definition of F , $F(y)$ codes γ_1 where γ_1 is the least element of E such that $\gamma_1 > \gamma_0$ and

$$(3.13) \quad (L_{\gamma_1}[A_0, E], E \cap \gamma_1) \prec (L_{\omega_2}[A_0, E], E).$$

By Claim 3.8, $F(y) \in L_{\alpha_0}[A]$. Since $F(y)$ codes γ_1 , $\gamma_1 < \alpha_0$. Since $\alpha_0 \leq \alpha$, $\gamma_1 < \alpha$. By (3.13) and (3.4), $\gamma_1 \leq \gamma_0$. Contradiction.

So the assumption that $\gamma_0 < \alpha$ is false. Then $\gamma_0 = \alpha$ and hence $\alpha \in E$. By (3.2) and Proposition 2.12, $L[G, H, A] \models \alpha$ has strong reflecting property. We have proved $L[G, H, A] \models (3.3)$.

Suppose $Y \prec L_{\alpha_A}[A]$, $|Y| = \omega$ and \bar{Y} is the transitive collapse of Y . Let $\bar{\omega}_1 = Y \cap \omega_1$. Then $\bar{Y} = L_{\bar{\alpha}}[\bar{A}]$ where $\bar{A} = A \cap \bar{\omega}_1$ and $\bar{\alpha} = o.t.(Y \cap \alpha_A)$. Note that $\bar{\omega}_1 < \omega_1$ and $L_{\bar{\alpha}}[\bar{A}] \models Z_3$. Suppose $\bar{\omega}_1 \leq \eta < \bar{\alpha}$ is \bar{A} -admissible. By (3.3), η is an L -cardinal. Let

$$Z = \{\delta < \omega_1 \mid \exists \alpha > \delta (L_\alpha[A \cap \delta] \models "Z_3 + \delta = \omega_1" \wedge \forall \eta ((\delta \leq \eta < \alpha \wedge \eta \text{ is } A \cap \delta\text{-admissible}) \rightarrow \eta \text{ is an } L\text{-cardinal}))\}.$$

⁴ $P = [\omega_1]^{<\omega_1} \times [Z_F]^{<\omega_1}$. $(p, q) \leq (p', q')$ iff $p \supseteq p', q \supseteq q'$ and $\forall \alpha \in q' (p \cap \delta_\alpha \subseteq p')$.

Let $Q = \{Y \cap \omega_1 \mid Y \prec L_{\alpha_A}[A] \wedge |Y| = \omega\}$. We have shown that $Q \subseteq Z$ and hence Z contains a club in ω_1 . Define

$$(3.14) \quad S = Z \cap \{\alpha < \omega_1 : \alpha \text{ is an } L\text{-cardinal}\}.$$

Then S is stationary and in fact contains a club.

3.3. Step Three. In this step, we shoot a club C through S via Baumgartner's forcing P_S^B such that $\text{Lim}(C)$ has property (3.30).⁵

We still work in $L[G, H, A]$. For $f \in P_S^B$, define $(P_S^B)_f = \{g \in P_S^B \mid g \leq f \text{ and } \max(\text{dom}(g)) = \max(\text{dom}(f))\}$. For $\eta < \omega_1$, define $P_S^B \restriction \eta = \{f \in P_S^B \mid (\text{dom}(f) \cup \text{ran}(f)) \subseteq \eta\}$.

Lemma 3.9. *Suppose $f \in P_S^B$ where $f = \{(\eta, \eta)\}$. Then*

$$(P_S^B)_f = \{g \cup \{(\eta, \eta)\} \mid g \in P_S^B \restriction \eta\}.$$

Proof. \subseteq is trivial. Fix $g \in P_S^B \restriction \eta$. We show that $g \cup \{(\eta, \eta)\} \in P_S^B$. It suffices to show that there exists $H : \eta + 1 \rightarrow S \cap (\eta + 1)$ such that

$$(3.15) \quad H \text{ is increasing and continuous, } H \text{ extends } g \text{ and } H(\eta) = \eta.$$

Let $\xi = \max(\text{dom}(g))$. Let $F : \xi + 1 \rightarrow S \cap (g(\xi) + 1)$ be the witness function for $g \in P_S^B$ (i.e. F is increasing, continuous and extends g). Let $E : \eta + 1 \rightarrow S \cap (\eta + 1)$ be the witness function for $f \in P_S^B$ (i.e. E is increasing, continuous and $E(\eta) = \eta$). Let $C = \text{ran}(E) \setminus (g(\xi) + 1)$. Since $\eta \in S$, η is indecomposable⁶ and hence $\text{o.t.}(C) = \text{o.t.}((\eta + 1) \setminus (g(\xi) + 1)) = \eta + 1$ since $g(\xi) < \eta$. Let $\pi : \eta + 1 \rightarrow C$ be an increasing continuous enumeration of C . Define $H : \eta + 1 \rightarrow S \cap (\eta + 1)$ by $H \restriction \xi + 1 = F$ and for any $\alpha \leq \eta$, $H(\xi + 1 + \alpha) = \pi(\alpha)$. It is easy to check that H satisfies (3.15). \square

Notation. For $\eta \in S$, let α_η be the least $\alpha > \eta$ such that $L_\alpha[A \cap \eta] \models Z_3 + \eta = \omega_1$.

Lemma 3.10. (1) *Suppose $\eta \in S$ and $\beta < \eta$. Then $L_\eta[A] \models \beta$ is countable.*

(2) *Suppose $\eta_0, \eta_1 \in S$ and $\eta_0 < \eta_1$. Then $\alpha_{\eta_0} < \eta_1$. i.e. For any $\eta \in S$, $\alpha_\eta < \bar{\eta}$ where $\bar{\eta} = \min(S \setminus (\eta + 1))$.*

Proof. (1) Since $\eta \in S$, $L_{\alpha_\eta}[A \cap \eta] \models \eta = \omega_1$. Note that $\mathbb{R} \cap L_{\alpha_\eta}[A \cap \eta] = \mathbb{R} \cap L_\eta[A \cap \eta] = \mathbb{R} \cap L_\eta[A]$. Since $\beta < \eta$, $L_\eta[A] \models \beta$ is countable.

(2) Suppose $\eta_1 \leq \alpha_{\eta_0}$. Note that $Z_3 \vdash \forall E \subseteq \omega_1 (L_{\omega_1}[E] \models ZFC^-)$. Since $L_{\alpha_{\eta_1}}[A \cap \eta_1] \models Z_3 + \eta_1 = \omega_1$, $L_{\eta_1}[A \cap \eta_0] \models ZFC^-$. Since $\eta_1 \leq \alpha_{\eta_0}$ and $L_{\alpha_{\eta_0}}[A \cap \eta_0] \models \eta_0 = \omega_1$, $L_{\eta_1}[A \cap \eta_0] \subseteq L_{\alpha_{\eta_0}}[A \cap \eta_0]$ and hence $L_{\eta_1}[A \cap \eta_0] \models \eta_0 = \omega_1$. Since $\eta_1 \in S$, $L_{\eta_1}[A \cap \eta_0] \models ZFC^-$, $L_{\eta_1}[A \cap \eta_0] \subseteq L_{\alpha_{\eta_0}}[A \cap \eta_0] \models Z_3$ and $L_{\eta_1}[A \cap \eta_0] \models \eta_0 = \omega_1$, we have $L_{\eta_1}[A \cap \eta_0] \models Z_3$. i.e.

$$(3.16) \quad L_{\eta_1}[A \cap \eta_0] \models Z_3 + \eta_0 = \omega_1.$$

So $\eta_1 \geq \alpha_{\eta_0}$ and hence $\eta_1 = \alpha_{\eta_0}$.

Fact 3.11. (Folklore, [8], [11]) $(Z_3) \quad \forall E \subseteq \omega_1 \forall \alpha < \omega_1 \forall a \in L_{\omega_1}[E] \exists X (X \prec L_{\omega_1}[E] \wedge |X| = \omega \wedge \alpha \cup \{a\} \subseteq X)$.

⁵We failed to shoot such a club via variants of Harrington's forcing. The key point is that Theorem 3.12 works for P_S^B but does not work for P_S .

⁶A limit ordinal γ is indecomposable if there is no $\alpha < \gamma$ and $\beta < \gamma$ such that $\alpha + \beta = \gamma$. Note that if γ is indecomposable, then for any $\alpha < \gamma$, $\text{o.t.}(\{\beta \mid \alpha \leq \beta < \gamma\}) = \gamma$.

Since $L_{\alpha_{\eta_1}}[A \cap \eta_1] \models Z_3 + \eta_1 = \omega_1$, by Fact 3.11, there is $X \in L_{\alpha_{\eta_1}}[A \cap \eta_1]$ such that $X \prec L_{\eta_1}[A \cap \eta_0]$, $L_{\alpha_{\eta_1}}[A \cap \eta_1] \models |X| = \omega$, $A \cap \eta_0 \in X$ and $\eta_0 + 1 \subseteq X$ (in Fact 3.11 let $E = A \cap \eta_0$, $\alpha = \eta_0 + 1$ and $a = A \cap \eta_0$). Let M be the transitive collapse of X and $M = L_{\bar{\eta}_1}[A \cap \eta_0]$. Note that $\eta_0 < \bar{\eta}_1 < \eta_1$. By (3.16), $L_{\bar{\eta}_1}[A \cap \eta_0] \models Z_3 + \eta_0 = \omega_1$ and hence $\alpha_{\eta_0} \leq \bar{\eta}_1 < \eta_1$. Contradiction. \square

Theorem 3.12. *Suppose $\{(\eta, \eta)\} \in P_S^B$. Then $(P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]} = P_S^B \restriction \eta$.*

Proof. \subseteq is trivial. Suppose $g \in P_S^B \restriction \eta$. We show that $g \in (P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]}$. Let $\xi = \max(\text{dom}(g))$. Let $H : \xi + 1 \rightarrow S \cap \eta$ be the witness function for $g \in P_S^B$ (i.e. H is increasing, continuous and extends g). It suffices to find a function $H^\infty : \xi + 1 \rightarrow S \cap \eta$ such that

$$(3.17) \quad H^\infty \text{ is increasing, continuous, } H^\infty \text{ extends } g \text{ and } H^\infty \in L_{\alpha_\eta}[A \cap \eta].$$

Pick a surjection $e_0 : \omega \rightarrow \xi + 1$ such that $e_0 \in L_{\alpha_\eta}[A \cap \eta]$ and

$$(3.18) \quad \text{for any } \alpha \leq \xi, \{i \in \omega \mid e_0(i) = \alpha\} \text{ is infinite.}$$

Pick a surjection $e_1 : \omega \rightarrow H(\xi) + 1$ such that $e_1 \in L_{\alpha_\eta}[A \cap \eta]$. Let T be the set of all pairs (π_1, π_2) such that $\pi_1 : k \rightarrow (H(\xi) + 1) \cap S$ where $k \in \omega$, $\pi_2 : k \rightarrow \omega$ and the following hold:

$$(3.19) \quad \text{For all } i < k, \text{ if } e_0(i) \in \text{dom}(g), \text{ then } \pi_1(i) = g(e_0(i));$$

$$(3.20) \quad \forall i < j < k (\pi_1(i) = \pi_1(j) \Leftrightarrow e_0(i) = e_0(j));$$

$$(3.21) \quad \forall i < j < k (\pi_1(i) < \pi_1(j) \Leftrightarrow e_0(i) < e_0(j));$$

$$(3.22) \quad \text{For all } i < k, \text{ if } e_0(i) > 0 \text{ is a limit ordinal and } \pi_2(i) < k, \text{ then} \\ \sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1(i)\}) < \pi_1(\pi_2(i)) < \pi_1(i) \text{ and } e_0(\pi_2(i)) < e_0(i).$$

By (3.14) and Lemma 3.10(2), $S \cap (H(\xi) + 1) \in L_{\alpha_\eta}[A \cap \eta]$. Since $g \in P_S^B \restriction \eta$, $g \in L_{\alpha_\eta}[A \cap \eta]$. Since $S \cap (H(\xi) + 1), g, e_0, e_1 \in L_{\alpha_\eta}[A \cap \eta]$, by the definition of T , $T \in L_{\alpha_\eta}[A \cap \eta]$.

Define $\pi_1^\infty : \omega \rightarrow (H(\xi) + 1) \cap S$ as follows: $\pi_1^\infty(i) = H(e_0(i))$ for $i \in \omega$. Now we define $\pi_2^\infty : \omega \rightarrow \omega$ as follows such that for all $i < \omega$, if $e_0(i) > 0$ is a limit ordinal, then

$$(3.23) \quad \sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1^\infty(i)\}) < \pi_1^\infty(\pi_2^\infty(i)) < \pi_1^\infty(i) \text{ and } e_0(\pi_2^\infty(i)) < e_0(i).$$

Suppose $e_0(i) > 0$ and $e_0(i)$ is a limit ordinal. Let $\alpha = e_0(i)$. Since H is continuous, $H(\alpha)$ is a limit ordinal. Let $\beta < \alpha$ be the least ordinal such that $\sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1^\infty(i)\}) < H(\beta) < H(\alpha)$. Let $\pi_2^\infty(i)$ be the least $j \in \omega$ such that $e_0(j) = \beta$. If $e_0(i) = 0$ or $e_0(i)$ is not a limit ordinal, let $\pi_2^\infty(i) = 0$. Since $\pi_1^\infty(\pi_2^\infty(i)) = \pi_1^\infty(j) = H(e_0(j)) = H(\beta)$, $\pi_1^\infty(i) = H(\alpha)$ and $e_0(\pi_2^\infty(i)) = \beta < \alpha = e_0(i)$, (3.23) holds.

Claim 3.13. For any $k \in \omega$, $(\pi_1^\infty \restriction k, \pi_2^\infty \restriction k) \in T$.

Proof. Fix $k \in \omega$. We show that $(\pi_1^\infty \restriction k, \pi_2^\infty \restriction k)$ satisfies conditions (3.19)-(3.22) in the definition of T . Since H extends g , (3.19) holds. Since H is strictly increasing, (3.20) and (3.21) hold. By (3.23), (3.22) holds.⁷ \square

⁷To show (3.23), we use that H is continuous.

Define $H^\infty : \xi + 1 \rightarrow S \cap \eta$ by

$$(3.24) \quad H^\infty(e_0(i)) = \pi_1^\infty(i) \text{ for } i \in \omega.$$

We show that H^∞ satisfies (3.17) via Claim 3.13. By (3.20), H^∞ is well defined. By (3.21), H^∞ is increasing. By (3.19), H^∞ extends g . Since $T, e_0 \in L_{\alpha_\eta}[A \cap \eta]$, by (3.24) and Claim 3.13, $H^\infty \in L_{\alpha_\eta}[A \cap \eta]$.

Claim 3.14. H^∞ is continuous.

Proof. Suppose $0 < \alpha \leq \xi$ is a limit ordinal. We show that $H^\infty(\alpha) = \sup(\{H^\infty(\beta) \mid \beta < \alpha\})$. Suppose not. Then there exists θ such that $\sup(\{H^\infty(\beta) \mid \beta < \alpha\}) < \theta < H^\infty(\alpha)$.

Pick m_0 such that $e_1(m_0) = \theta$. By (3.18), pick $i > m_0$ such that $e_0(i) = \alpha$. Since $e_1(m_0) = \theta < H^\infty(\alpha)$, by (3.24), $\theta \leq \sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1^\infty(i)\})$. By (3.24), $\pi_1^\infty(\pi_2^\infty(i)) = H^\infty(e_0(\pi_2^\infty(i)))$. By (3.23), $\theta < H^\infty(e_0(\pi_2^\infty(i)))$ and $e_0(\pi_2^\infty(i)) < e_0(i) = \alpha$. But $\sup(\{H^\infty(\beta) \mid \beta < \alpha\}) < \theta$. Contradiction.⁸ \square

\square

Theorem 3.15. Suppose $f \in P_S^B$ where $f = \{(\eta, \eta)\}$. Then

$$(P_S^B)_f = \{g \cup \{(\eta, \eta)\} \mid g \in (P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]}\}.$$

Proof. Follows from Lemma 3.9 and Theorem 3.12. \square

Suppose G^* is P_S^B -generic over $L[G, H, A]$. Define $F_{G^*} = \bigcup\{f \mid f \in G^*\}$. Then $F_{G^*} : \omega_1 \rightarrow S$ is increasing and continuous. Let $C = \text{ran}(F_{G^*})$. Then $C \subseteq S$ is a club in ω_1 . Let $\text{Lim}(C) = \{\alpha \mid \alpha \text{ is a limit point of } C\}$. Now we work in $L[G, H, A, C]$.

Fact 3.16. (Folklore, [17]) Suppose $M \models Z_3, P \in M$ is a forcing notion, $M \models |P| \leq \omega_1$ and G is P -generic over M . If $M \models P$ preserves ω_1 , then $M[G] \models Z_3$.

Theorem 3.17. Suppose $\eta \in \text{Lim}(C)$. Then

$$L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1 \Leftrightarrow L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3.$$

Proof. (\Rightarrow) Suppose $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$. Then

$$(3.25) \quad L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models C \cap \eta \text{ is a club in } \eta.$$

We show that

$$(3.26) \quad L_{\alpha_\eta}[A \cap \eta] \models S \cap \eta \text{ is stationary.}$$

Suppose not. Then there exists a club E in η such that $E \in L_{\alpha_\eta}[A \cap \eta]$ and $E \cap S \cap \eta = \emptyset$. Then $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models E$ and $C \cap \eta$ are disjoint closed subsets of η . Contradiction.

By (3.25), *o.t.* $(C \cap \eta) = \eta$ and hence η is the η -th element of C . Since $F_{G^*}(\xi)$ is the ξ -th element of C , $F_{G^*}(\eta) = \eta$. Let $f = \{(\eta, \eta)\}$. Since $f \in G^*$, by Lemma 2.7, $G^* \cap (P_S^B)_f$ is $(P_S^B)_f$ -generic over V . By Theorem 3.15, $(P_S^B)_f = \{h \cup \{(\eta, \eta)\} \mid h \in (P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]}\}$. So $G^* \cap (P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]}$ is $(P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]}$ -generic over $L_{\alpha_\eta}[A \cap \eta]$ and hence

$$(3.27) \quad C \cap \eta \text{ is } (P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]} \text{-generic over } L_{\alpha_\eta}[A \cap \eta].$$

⁸The proof of Theorem 3.12 depends on (3.14) and property of Baumgartner's forcing. In fact, its proof only uses the part $(\forall \eta \in S)(\exists \delta > \eta(L_\delta[A \cap \eta] \models Z_3 + \eta = \omega_1))$ in (3.14).

By (3.26), do Baumgartner's forcing $P_{S \cap \eta}^B$ over $L_{\alpha_\eta}[A \cap \eta]$. Since $L_{\alpha_\eta}[A \cap \eta] \models Z_3$, by Fact 2.6, $L_{\alpha_\eta}[A \cap \eta] \models |(P_{S \cap \eta}^B)| = \omega_1$ and $P_{S \cap \eta}^B$ preserves ω_1 . By (3.27) and Fact 3.16, $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$.

(\Leftarrow) Suppose $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$. We show that $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$. Suppose not. i.e. $\eta < \omega_1^{L_{\alpha_\eta}[A \cap \eta, C \cap \eta]}$. Since $L_{\alpha_\eta}[A \cap \eta] \subseteq L_{\alpha_\eta}[A \cap \eta, C \cap \eta]$, $\omega_1^{L_{\alpha_\eta}[A \cap \eta, C \cap \eta]}$ is a cardinal in $L_{\alpha_\eta}[A \cap \eta]$. But since $L_{\alpha_\eta}[A \cap \eta] \models Z_3 + \eta = \omega_1$, $\eta = \omega_1^{L_{\alpha_\eta}[A \cap \eta]}$ is the largest cardinal in $L_{\alpha_\eta}[A \cap \eta]$. Contradiction.⁹ \square

As a summary, by (3.14) and Theorem 3.17, $Lim(C)$ has the following properties:

$$(3.28) \quad \forall \eta \in Lim(C) (\eta \text{ is an } L\text{-cardinal});$$

$$(3.29) \quad \forall \eta \in Lim(C) ((\eta \leq \beta < \alpha_\eta \wedge \beta \text{ is } A \cap \eta\text{-admissible}) \rightarrow \beta \text{ is an } L\text{-cardinal});$$

$$(3.30) \quad \forall \eta \in Lim(C) (L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1 \rightarrow L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3).$$

3.4. Step Four. In this step, we use properties of $Lim(C)$ to define the almost disjoint system on ω and some $B^* \subseteq \omega_1$. Then we do almost disjoint forcing to code B^* by a real x . Finally, we use (3.28)-(3.30) to show that x is the witness real for HP.

We still work in $L[G, H, A, C]$. Take α and X such that $L_\alpha[A] \models Z_3$, $X \prec L_\alpha[A, C]$, $|X| = \omega$ and $X \cap \omega_1 \in Lim(C)$.¹⁰ Let $\eta = X \cap \omega_1$. The transitive collapse of X is in the form $L_{\bar{\alpha}}[A \cap \eta, C \cap \eta]$. Note that $L_{\bar{\alpha}}[A \cap \eta] \models Z_3$ and

$$(3.31) \quad L_{\bar{\alpha}}[A \cap \eta, C \cap \eta] \models \eta = \omega_1.$$

By (3.31), $L_{\bar{\alpha}}[A \cap \eta] \models \eta = \omega_1$. So $\alpha_\eta \leq \bar{\alpha}$. By (3.31), $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$. Since $\eta \in Lim(C)$, by (3.30), $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$. Let

$$(3.32) \quad \eta^* \text{ be the least } \eta \in Lim(C) \text{ such that } L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3 + \eta = \omega_1.$$

Note that η^* is a limit point of $Lim(C)$.¹¹

Lemma 3.18. *Suppose $\eta \in Lim(C)$, $\eta < \eta^*$ and $\beta < \alpha_\eta$. Then $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \beta < \omega_1$.*

Proof. Since $L_{\alpha_\eta}[A \cap \eta] \models Z_3$, $L_{\alpha_\eta}[A \cap \eta] \models \forall \beta \in Ord (|\beta| \leq \omega_1)$. Since $L_{\alpha_\eta}[A \cap \eta] \models \eta = \omega_1$ and $\beta < \alpha_\eta$, there exists $f \in L_{\alpha_\eta}[A \cap \eta]$ such that $f : \eta \rightarrow \beta$ is surjective.

Claim 3.19. $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta < \omega_1$.

Proof. Suppose $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$. By (3.30), $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$. By (3.32), $\eta \geq \eta^*$. Contradiction. \square

So there exists $g \in L_{\alpha_\eta}[A \cap \eta, C \cap \eta]$ such that $g : \omega \rightarrow \eta$ is surjective. So $f \circ g : \omega \rightarrow \beta$ is surjective and $f \circ g \in L_{\alpha_\eta}[A \cap \eta, C \cap \eta]$. Hence $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \beta < \omega_1$. \square

⁹The key step in Theorem 3.17 is to show that (3.26) implies (3.27) which depends on the representation theorem for $(P_{S \cap \eta}^B)^{L_{\alpha_\eta}[A \cap \eta]}$ (Theorem 3.12).

¹⁰By Fact 3.16, such α exists.

¹¹Suppose not. Let $\xi < \eta^*$ be the largest element of $Lim(C)$. Then $o.t.(C \cap (\eta^* \setminus (\xi + 1))) = \omega$. But since $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*] \models \eta^* = \omega_1$, $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*] \models C \cap \eta^*$ is a club in η^* . Contradiction.

Now we work in $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*]$. We first define an almost disjoint system $\langle \delta_\beta : \beta < \eta^* \rangle$ on ω and $B^* \subseteq \eta^*$. To define $\langle \delta_\beta : \beta < \eta^* \rangle$ we first define $\langle f_\beta : \beta < \eta^* \rangle$ by induction on $\beta < \eta^*$. If $\beta < \omega$, let $f_\beta : \omega \rightarrow 1 + \beta$ be a recursive function. Fix $\omega \leq \beta < \eta^*$. Let $\eta_0 = \sup(\text{Lim}(C) \cap \beta)$ and $\eta_1 = \min(C \setminus (\beta + 1))$.

- Definition 3.20.** (1) Suppose $\eta_0 = 0$. Since $\eta_1 \in C$ and $\beta < \eta_1$, by Lemma 3.10(1), $L_{\eta_1}[A] \models \beta$ is countable. Let $f_\beta : \omega \rightarrow \beta$ be the least surjection in $L_{\eta_1}[A]$.
 (2) Suppose $\eta_0 \neq 0$ and $\beta < \alpha_{\eta_0}$. Since $\eta_0 \in \text{Lim}(C)$, $\eta_0 < \eta^*$ and $\beta < \alpha_{\eta_0}$, by Lemma 3.18, $L_{\alpha_{\eta_0}}[A \cap \eta_0, C \cap \eta_0] \models \beta < \omega_1$. Let $f_\beta : \omega \rightarrow \beta$ be the least surjection in $L_{\alpha_{\eta_0}}[A \cap \eta_0, C \cap \eta_0]$.
 (3) Suppose $\eta_0 \neq 0$ and $\beta \geq \alpha_{\eta_0}$. Since $\eta_1 \in S$ and $\beta < \eta_1$, by Lemma 3.10(1), $L_{\eta_1}[A] \models \beta$ is countable. Let $f_\beta : \omega \rightarrow \beta$ be the least surjection in $L_{\eta_1}[A]$.

Now we define an almost disjoint system $\langle \delta_\beta : \beta < \eta^* \rangle$ on ω from $\langle f_\beta : \beta < \eta^* \rangle$. Fix a recursive bijection $\pi : \omega \leftrightarrow \omega \times \omega$. Let $x_\beta = \{(i, j) \mid f_\beta(i) < f_\beta(j)\}$ and $y_\beta = \{k \in \omega \mid \pi(k) \in x_\beta\}$. Let $\langle s_i \mid i \in \omega \rangle$ be an injective, recursive enumeration of $\omega^{<\omega}$ and $\delta_\beta = \{i \in \omega \mid \exists m \in \omega (s_i = y_\beta \cap m)\}$. Then $\langle \delta_\beta : \beta < \eta^* \rangle$ is a sequence of almost disjoint reals. Since $\langle s_i \mid i \in \omega \rangle$ is recursive, π is recursive and for any $i \in \omega$, f_i is recursive, $\langle \delta_i : i \in \omega \rangle$ is recursive.

Now we define $B^* \subseteq \eta^*$. Fix $\beta < \eta^*$. We define z_β as follows. Let

$$(3.33) \quad \eta_0^\beta = \min(\text{Lim}(C) \setminus (\beta + 1)) \text{ and } \eta_1^\beta = \min(\text{Lim}(C) \setminus (\eta_0^\beta + 1)).$$

Note that $\eta_1^\beta < \eta^*$ since η^* is a limit point of $\text{Lim}(C)$. By Lemma 3.10(2), $\alpha_{\eta_0^\beta} < \alpha_{\eta_1^\beta}$. By Lemma 3.18, $\alpha_{\eta_0^\beta}$ is countable in $L_{\alpha_{\eta_1^\beta}}[A \cap \eta_1^\beta, C \cap \eta_1^\beta]$. Let z_β be the least real in $L_{\alpha_{\eta_1^\beta}}[A \cap \eta_1^\beta, C \cap \eta_1^\beta]$ such that

$$(3.34) \quad z_\beta \text{ codes } \langle \eta_0^\beta, \alpha_{\eta_0^\beta}, A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle.$$

$$(3.35) \quad \text{Define } B^* = \{\omega \cdot \alpha + i \mid \alpha < \eta^* \wedge i \in z_\alpha\}.$$

By almost disjoint forcing, we get a real x such that for $\alpha < \eta^*$,

$$(3.36) \quad \alpha \in B^* \Leftrightarrow |x \cap \delta_\alpha| < \omega.$$

Since $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*] \models Z_3$ and x is a generic real built via a c.c.c forcing, by Fact 3.16, $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*][x] \models Z_3$. By (3.36), (3.35) and (3.34), x codes $(A \cap \eta^*, C \cap \eta^*)$ via $\langle \delta_\beta : \beta < \eta^* \rangle$. So $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*][x] = L_{\alpha_{\eta^*}}[x]$ and hence $L_{\alpha_{\eta^*}}[x] \models Z_3$.

We want to show that $L_{\alpha_{\eta^*}}[x] \models \text{HP}$. By absoluteness, it suffices to show in $L[G, H, A, C, x]$ that if $\lambda < \alpha_{\eta^*}$ is x -admissible, then λ is an L -cardinal. Now we work in $L[G, H, A, C, x]$. In the rest of this section, we fix $\lambda < \alpha_{\eta^*}$ and assume that

$$(3.37) \quad \lambda \text{ is } x\text{-admissible.}$$

Since $\langle \delta_i \mid i \in \omega \rangle$ is recursive, by (3.37), $\langle \delta_i \mid i \in \omega \rangle \in L_\lambda[x]$. By (3.35), $B^* \cap \omega = z_0$. By (3.36), $B^* \cap \omega = \{i \in \omega \mid |x \cap \delta_i| < \omega\}$. By (3.37), $z_0 \in L_\lambda[x]$.

Definition 3.21. $\theta = \sup(\{\beta < \eta^* \mid z_\beta \in L_\lambda[x]\})$ and $\gamma = \sup(\{\eta_0^\beta \mid \beta < \theta\})$.

By (3.33) and (3.34), for $\beta < \eta^*$, $z_\beta = z_{\beta+1}$. So θ is a limit ordinal. By (3.34), if $\beta_0 < \beta_1 < \eta^*$, then z_{β_0} is recursive in z_{β_1} . So if $\beta < \theta$, then by (3.37), $z_\beta \in L_\lambda[x]$.

Lemma 3.22. $\langle z_\beta \mid \beta < \theta \rangle$ is Σ_1 -definable in $L_\lambda[x]$ from x .

Proof. If $\beta < \theta$, then $z_\beta \in L_\lambda[x]$ and hence by (3.34) and (3.37), there exists $\lambda_0 < \lambda$ such that λ_0 is a limit ordinal and $\langle \eta_0^\beta, \alpha_{\eta_0^\beta}, A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle \in L_{\lambda_0}[x]$. We can find a formula $\varphi(\alpha, z, \beta, x, A, C)$ which says that $\langle \eta_0^\beta, \alpha_{\eta_0^\beta}, A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle$ is countable in $L_\alpha[x]$ and z is the $<_{L_\alpha[x]}$ -least real which codes $\langle \eta_0^\beta, \alpha_{\eta_0^\beta}, A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle$.

By absoluteness, for $\beta < \theta$, $z = z_\beta$ if and only if $\exists \lambda_0 < \lambda (z \in L_{\lambda_0}[x] \wedge \lambda_0 \text{ is a limit ordinal} \wedge L_{\lambda_0}[x] \models \varphi[\lambda_0, z, \beta, x, A, C])$. \square

Theorem 3.23. λ is an L -cardinal.

Proof. If $\beta < \theta$, then since z_β codes η_0^β and $z_\beta \in L_\lambda[x]$, by (3.37), $\beta < \eta_0^\beta < \lambda$. Hence $\theta \leq \lambda$ and $\gamma \leq \lambda$.

Case 1: $\theta = \lambda$. Then $\gamma = \sup(\{\eta_0^\beta \mid \beta < \lambda\}) = \lambda$. Since $\gamma \in \text{Lim}(C)$, by (3.28), λ is an L -cardinal.

Case 2: $\theta < \lambda$. By Lemma 3.22 and (3.37), $\langle z_\beta \mid \beta < \theta \rangle \in L_\lambda[x]$. Note that z_β codes $(A \cap \eta_0^\beta, C \cap \eta_0^\beta)$ for $\beta < \theta$ and hence $(A \cap \gamma, C \cap \gamma) \in L_\lambda[x]$.

Subcase 1: $\alpha_\gamma \leq \lambda$. Since $\gamma, \eta^* \in \text{Lim}(C)$ and $\lambda < \alpha_{\eta^*}$, by Lemma 3.10(2), $\gamma < \eta^*$. For $i \in \omega$, since $\gamma + i < \alpha_\gamma$, by Definition 3.20(2), $f_{\gamma+i} : \omega \rightarrow \gamma + i$ is the least surjection in $L_{\alpha_\gamma}[A \cap \gamma, C \cap \gamma]$.¹² So $\langle \delta_{\gamma+i} \mid i \in \omega \rangle$ is Σ_1 -definable in $L_{\alpha_\gamma}[A \cap \gamma, C \cap \gamma]$ from $(A \cap \gamma, C \cap \gamma)$. Since $(A \cap \gamma, C \cap \gamma) \in L_\lambda[x]$, $\langle \delta_{\gamma+i} \mid i \in \omega \rangle$ is Σ_1 -definable in $L_\lambda[x]$ from x . By (3.37), $\langle \delta_{\gamma+i} \mid i \in \omega \rangle \in L_\lambda[x]$. Note that $\omega \cdot \gamma = \gamma$ and $z_\gamma = \{i \in \omega \mid \omega \cdot \gamma + i \in B^*\} = \{i \in \omega \mid |x \cap \delta_{\gamma+i}| < \omega\}$. By (3.37), $z_\gamma \in L_\lambda[x]$ and hence $\gamma < \theta$. By the definition of $\gamma, \eta_0^\gamma \leq \gamma$. Contradiction.

Subcase 2: $\lambda < \alpha_\gamma$. Since $A \cap \gamma \in L_\lambda[x]$, by (3.37), λ is $A \cap \gamma$ -admissible. Since $\gamma \in \text{Lim}(C)$ and $\gamma \leq \lambda < \alpha_\gamma$, by (3.29), λ is an L -cardinal. \square

So $L_{\alpha_{\eta^*}}[x] \models Z_3 + \text{HP}$ and we have proved our main Theorem 1.5.¹³ As a corollary, $Z_3 + \text{HP}$ does not imply 0^\sharp exists.¹⁴ But $Z_4 + \text{HP}$ implies 0^\sharp exists which we construe as part of the folklore, cf.[5]. So Z_4 is the minimal system in higher order arithmetic to show that HP implies 0^\sharp exists.

Our proof of the main Theorem 1.5 shows that if we can force a club in ω_2 of L -cardinals with weakly reflecting property via set forcing, then we can force a set model of $Z_3 + \text{HP}$ via set forcing without reshaping. In our proof, the assumption “there exists a remarkable cardinal with a weakly inaccessible cardinal above it” is only used in Step One to force a club in ω_2 of L -cardinals with weakly reflecting property.

We conclude this section with a remark about the amount of strong reflecting property needed in our proof. For our proof, we need that ω_2 has strong reflecting property. Only knowing that some $\gamma \in [\omega_1, \omega_2)$ has strong reflecting property is not enough for our proof.

REFERENCES

- [1] J. Barwise. *Admissible Sets and Structures*. Perspectives in Math. Logic Vol.7, Springer Verlag, 1976.

¹²This is the place we use (3.30): Definition 3.20(2) uses Lemma 3.18 which follows from (3.30).

¹³To define an almost disjoint system on ω , we usually use the reshaping technique. However, in our proof we did not use reshaping and instead we use properties of $\text{Lim}(C)$ to define the almost disjoint system.

¹⁴From [15], any remarkable cardinal is remarkable in L .

- [2] James E. Baumgartner. Applications of the Proper Forcing Axiom, Handbook of set-theoretic topology (K. Kunen and J. E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp.913-959.
- [3] Yong Cheng. Note on Hugh Woodin's proof of Harrington's theorem.
- [4] Keith J. Devlin. *Constructibility*. Springer, Berlin, 1984.
- [5] L.A. Harrington. Analytic determinacy and 0^\sharp . *The Journal of Symbolic Logic*, 43(1978), 685-693.
- [6] L.A. Harrington, James E. Baumgartner and E.M. Kleinberg. Adding a closed unbounded set. *The Journal of Symbolic Logic*, 41, 481-482, 1976.
- [7] Thomas J. Jech. *Multiple forcing*. Cambridge University Press 1986.
- [8] Thomas J. Jech. *Set Theory, Third millennium edition, revised and expanded*. Springer, Berlin, 2003.
- [9] R.B. Jensen and R.M. Solovay. *Some applications of almost disjoint sets*. Mathematical Logic and Foundations of Set Theory, Proceedings of an International Colloquium Held Under the Auspices of The Israel Academy of Sciences and Humanities, Volume 59, 1970, Pages 84-104.
- [10] Akihiro Kanamori. *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*. Springer Monographs in Mathematics, Springer, Berlin, 2003, Second edition.
- [11] Kenneth Kunen. *Set Theory: An Introduction to Independence Proofs*. North Holland, 1980.
- [12] R. Mansfield and G. Weitkamp. *Recursive Aspects of Descriptive Set Theory*. Oxford Univ. Press, Oxford, 1985.
- [13] Yiannis N. Moschovakis. *Descriptive Set Theory*. North-Holland, Amsterdam, 1980.
- [14] Ramez L. Sami. Analytic determinacy and 0^\sharp : A forcing-free proof of Harrington's theorem. *Fundamenta Mathematicae*, 160(1999).
- [15] Ralf Schindler. Proper forcing and remarkable cardinals II. *Journal of Symbolic Logic*, 66 (2001), pp. 1481- 1492.
- [16] Ralf Schindler. *Set theory: exploring independence and truth*. Springer-Verlag 2014, to appear.
- [17] W. Hugh Woodin. Personal communication from W. Hugh Woodin.

INSTITUT FÜR MATHEMATISCHE LOGIK UND GRUNDLAGENFORSCHUNG, FACHBEREICH MATHEMATIK UND INFORMATIK, UNIVERSITÄT MÜNSTER, EINSTEINSTR. 62, 48149 MÜNSTER, GERMANY
E-mail address: world-cyr@hotmail.com